Discrete Exponent Function (1/14)

The Discrete Exponent Function (**DEF**) used in cryptography firstly was introduced in the cyclic multiplicative group $Z_p^* = \{1, 2, 3, \ldots, p\text{-}1\}$, with binary multiplication operation *** mod** p, where p is prime number. Further the generalizations were made especially in *Elliptic Curve Groups* laying a foundation of *Elliptic Curve CryptoSystems* (ECCS) in general and in *Elliptic Curve Digital Signature Algorithm* (ECDSA) in particular.

Let g be a generator of Z_p^* then **DEF** is defined in the following way:

$$\mathbf{DEF}_g(x) = g^x \mathbf{\ mod\ } p = a;$$

DEF argument x is associated with the private key – **PrK** (or other secret parameters) and therefore we will label it in red and value a is associated with public key – **PuK** (or other secret parameters) and therefore we will label it in green.

In order to ensure the security of cryptographic protocols, a large prime number p is chosen. This prime number has a length of 2048 bits, which means it is represented in decimal as being on the order of 2^{2048}, or approximately $p \sim 2^{2048}$.

In our modeling with Octave, we will use p of length having only 28 bits for convenience. We will deal also with a strong prime numbers.

Discrete Exponent Function (2/14)

Definition. Binary operation *** mod** p in Z_p^* is an arithmetic multiplication of two integers called operands and taking the result as a residue by dividing by p.

For example, let $p = 11$, then $Z_{11}^* = \{1, 2, 3, \ldots, 10\}$, then $5 * 8$ **mod** $11 = 40$ **mod** $11 = 7$, where $7 \in Z_p^*$.

In our example the residue of 40 by dividing by 11 is equal to 7, i.e., $40 = 3 * 11 + 7$.
Then 40 **mod** 11 = (33 + 7) **mod** 11 = (33 **mod** 11 + 7 **mod** 11) **mod** 11 = (0 + 7) **mod** 11 = 7.
Notice that 33 **mod** 11 = 0 and 7 **mod** 11 = 7.

Definition: The integer g is a generator in Z_p^* if powering it by integer exponent values x all obtained numbers that are computed **mod** p generates all elements in in Z_p^*.

So, it is needed to have at least p-1 exponents x to generate all p-1 elements of Z_p^*. You will see that exactly p-1 exponents x is enough.

Discrete Exponent Function (3/14)

Let Γ be the set of generators in Z_p^*. How to find a generator in Z_p^*?

In general, it is a hard problem, but using strong prime p and *Lagrange theorem in group theory* the generator in Z_p^* can be found by random search satisfying two following conditions.

For all $g \in \Gamma$

$$g^q \neq 1 \text{ mod } p; \text{ and } g^2 \neq 1 \text{ mod } p.$$

Fermat little theorem: If p is prime then for all integers n:

$$i^{p-1} = 1 \text{ mod } p.$$

Corollaries: 1. The exponent p-1 is equivalent to the exponent 0, since $i^0 = i^{p-1} = 1 \text{ mod } p$.

2. Any exponent e can be reduced **mod** (p-1), i.e.

$$i^e \text{ mod } p = n^{e \text{ mod } (p-1)} \text{ mod } p.$$

3. All non-equivalent exponents x are in the set $Z_{p-1} = \{0, 1, 2, \ldots, p-2\}$.

4. Sets Z_{p-1} and Z_p^* have the same number of elements.

Discrete Exponent Function (4/14)

In Z_{p-1} addition +, multiplication $*$ and subtraction - operations are realized **mod** (p-1).

Subtraction operation $(h-d)$ **mod** (p-1) is replaced by the following addition operation $(h + (-d))$ **mod** (p-1)).

Therefore, it is needed to find $-d$ **mod** (p-1) such that $d + (-d) = 0$ **mod** (p-1), then assume that

$$-d \text{ mod } (p-1) = (p-1-d).$$

Indeed, according to the distributivity property of modular operation

$$(d + (-d)) \text{ mod } (p-1) = (d + (p-1-d) \text{ mod } (p-1) = (p-1) \text{ mod } (p-1) = 0.$$

Then

$$(h-d) \text{ mod } (p-1) = (h + (p-1-d)) \text{ mod } (p-1)$$

Discrete Exponent Function (5/14)

Statement: If greatest common divider between p-1 and i is equal to 1, i.e., gcd(p-1, i) = 1, then there exists unique inverse element i^{-1} **mod** (p-1) such that $i * i^{-1}$ **mod** (p-1) = 1. This element can be found by *Extended Euclide algorithm* or using *Fermat little theorem*. We do not fall into details how to find i^{-1} **mod** (p-1) since we will use the ready-made computer code instead in our modeling.

Division operation / **mod** (p-1) of any element in Z_{p-1} by some element i is replaced by multiplication * operation with i^{-1} **mod** (p-1) if gcd(i, p-1) = 1 according to the *Statement* above.

To compute u/i **mod** (p-1) it is replaced by the following relation $u * i^{-1}$ **mod** (p - 1) since

$$u / i \bmod (p-1) = u * i^{-1} \bmod (p-1).$$

Discrete Exponent Function (6/14)

Example 1: Let for given integers u, x and h in Z_{p-1} we compute exponent s of generator g by the expression

$$s = u + xh.$$

Then

$$g^s \bmod p = g^{s \bmod (p-1)} \bmod p.$$

Therefore, s can be computed **mod** (p-1) in advance, to save a multiplication operations, i.e.

$$s = u + xh \bmod (p-1).$$

Example 2: Exponent s computation including subtraction by xr **mod** (p-1) and division by i in Z_{p-1} when gcd(i, p-1) = 1.

$$s = (h - xr)i^{-1} \bmod (p-1).$$

Firstly $d = xr$ **mod** (p-1) is computed:

Secondly $-d = -xr$ **mod** (p-1) = (p-1-d) is found.

Thirdly i^{-1} **mod** (p-1) is found.

And finally exponent $s = (h + (p\text{-}1\text{-}d))i^{-1}$ **mod** (p-1) is computed.

Discrete Exponent Function (7/14)

Referencing to Fermat little theorem and its corollaries, formulated above, the following theorem can be proved.

Theorem. If g is a generator in Z_p^* then **DEF** provides the following 1-to-1 mapping

$$\textbf{DEF}: Z_{p-1} \rightarrow Z_p^*.$$

Parameters p and g for **DEF** definition we name as Public Parameters and denote by **PP** = (p, g).

Example: Strong prime p = 11, p = 2 * 5 + 1, then q = 5 and q is prime. Then p-1 = 10.

$$Z_{11}^* = \{1, 2, 3, \ldots, 10\}$$

$$Z_{10} = \{0, 1, 2, \ldots, 9\}$$

Discrete Exponent Function (8/14)

The results of any binary operation (multiplication, addition, etc.) defined in any finite group is named *Cayley table* including multiplication table, addition table etc.

Multiplication table of multiplicative group Z_{11}^* is represented below.

$Z_{11}^* = \{1, 2, 3, ..., 10\}$ * mod 11

Multiplication tab. mod 11	***Z*11***									
*	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	1	3	5	7	9
3	3	6	9	1	4	7	10	2	5	8
4	4	8	1	5	9	2	6	10	3	7
5	5	10	4	9	3	8	2	7	1	6
6	6	1	7	2	8	3	9	4	10	5
7	7	3	10	6	2	9	5	1	8	4
8	8	5	2	10	7	4	1	9	6	3
9	9	7	5	3	1	10	8	6	4	2
10	10	9	8	7	6	5	4	3	2	1

Values of inverse elements in *Z*11*

1^{-1} = 1 mod 11
2^{-1} = 6 mod 11
3^{-1} = 4 mod 11
4^{-1} = 3 mod 11
5^{-1} = 9 mod 11
6^{-1} = 2 mod 11
7^{-1} = 8 mod 11
8^{-1} = 7 mod 11
9^{-1} = 5 mod 11
10^{-1} = 10 mod 11

12 | 11
-11 | 1
1

2/2 mod 11 =

$2 * 2^{-1}$ mod 11 = 1

2^{-1} mod 11 = 6.

5^{-1} mod 11 = 9

>> mulinv(5, 11) = 9

>> mod(5*9, 11) = 1

Discrete Exponent Function (9/14)

DEF$_g(x) = g^x$ **mod** $p = a$;

The table of exponent values for p = 11 in Z_{11}^* computed **mod** 11 and is presented in table below.

Notice that according to Fermat little theorem for all $z \in Z_{11}^*$, $z^{p-1} = z^{10} = z^0 = 1$ **mod** 11.

$p - 1 = 10$

$Z_{11}^* = \{1, 2, 3, ..., 10\}$

Exponent tab. mod 11	***Z*11***										
^	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	*10*
1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	4	8	5	10	9	7	3	6	1
3	1	3	9	5	4	1	3	9	5	4	1
4	1	4	5	9	3	1	4	5	9	3	1
5	1	5	3	4	9	1	5	3	4	9	1
6	1	6	3	7	9	10	5	8	4	2	1
7	1	7	5	2	3	10	4	6	9	8	1
8	1	8	9	6	4	10	3	2	5	7	1
9	1	9	4	3	5	1	9	4	3	5	1
10	1	10	1	10	1	10	1	10	1	10	1

List of generators when $q = 5$

$2^2 \neq 1$ **mod** 11 & $2^5 \neq 1$ **mod** 11
$6^2 \neq 1$ **mod** 11 & $6^5 \neq 1$ **mod** 11
$7^2 \neq 1$ **mod** 11 & $7^5 \neq 1$ **mod** 11
$8^2 \neq 1$ **mod** 11 & $8^5 \neq 1$ **mod** 11

$\Gamma = \{2, 6, 7, 8\}$

40% generators in Z_{10}^*

$p \simeq 2^{2048}$

$|p| \simeq 2048$

$Z_p^* = \{1, 2, 3, ..., p-1\}$

Let Γ be the set of generators in Z_p^*. How to find a generator in Z_p^*?

In general, it is a hard problem, but using strong prime p and *Lagrange theorem in group theory* the generator in Z_p^* can be found by random search satisfying two following conditions.

if p is a strong prime.
p is a strong prime if p is prime and $p = 2 \cdot q + 1$, when q - is prime

For all $g \in \Gamma$; choose >> g = randi(p)

$$g^q \neq 1 \textbf{ mod } p; \text{ and } g^2 \neq 1 \textbf{ mod } p.$$

Fermat little theorem: If p is prime then for all integers i:

$$i^{p-1} = 1 \textbf{ mod } p.$$

Corollaries: 1. The exponent p-1 is equivalent to the exponent 0, since $i^0 = i^{p-1} = 1$ mod p.   >> p = genstrongprime(28)

2. Any exponent e can be reduced **mod** (p-1), i.e.

$$i^e \textbf{ mod } p = i^{e \textbf{ mod } (p-1)} \textbf{ mod } p.$$

3. All non-equivalent exponents x are in the set $Z_{p-1} = \{0, 1, 2, \ldots, p\text{-}2\}$.

4. Sets Z_{p-1} and Z_p^* have the same number of elements.

The set of exponents x of DEF $\textbf{DEF}_g(x) = g^x \textbf{ mod } p = a$;

$+, -, *$ mod $_{(p-1)}$ and $/$ mod p (with exception) $+, -, *$ mod 10 and $/$ mod 10

$Z_{p-1} = \{0, 1, 2, \ldots, p-2\}$; when $p = 11$ ➡ $Z_{10} = \{0, 1, 2, \ldots, 9\}$

The set of values of DEF $\textbf{DEF}_g(x) = g^x \textbf{ mod } p = a$;

$Z_p^* = \{1, 2, 3, \ldots, p-1\}$

$|Z_{p-1}| = |Z_p^*| = p-1$; $p = 11$ ➡ $|Z_{10}| = |Z_{11}^*|$

Corollary: DEF provides a 1-to-1 atvaizdavimas if g is generator.

$$\text{DEF}_g : \underset{x}{Z_{p-1}} \longleftrightarrow \underset{a}{Z_p^*}$$

$p = 11$ is a strong prime since $p = 2 \cdot 5 + 1$
$q = 5$ is prime

Discrete Exponent Function (10/14)

Notice that there are elements satisfying the following different relations, for example:

$$3^5 = 1 \textbf{ mod } 11 \text{ and } 3^2 \neq 1 \textbf{ mod } 11.$$

The set of such elements forms a subgroup of prime order $q = 5$ if we add to these elements the *neutral group element* 1.

This subgroup has a great importance in cryptography we denote by

$$G_5 = \{1, 3, 4, 5, 9\}.$$

The multiplication table of G_5 elements extracted from multiplication table of Z_{11}^* is presented below.

Multiplication tab. mod 11	***G5***				
*	**1**	**3**	**4**	**5**	**9**
1	1	3	4	5	9
3	3	9	1	4	5
4	4	1	5	9	3
5	5	4	9	3	1
9	9	5	3	1	4

Values of inverse elements in G_5
$1^{-1} = 1$ mod 11
$3^{-1} = 4$ mod 11
$4^{-1} = 3$ mod 11
$5^{-1} = 9$ mod 11
$9^{-1} = 5$ mod 11

Exponent tab. mod 11	***G5***					
^	**0**	**1**	**2**	**3**	**4**	**5**
1	1	1	1	1	1	1
3	1	3	9	5	4	1
4	1	4	5	9	3	1
5	1	5	3	4	9	1
9	1	9	4	3	5	1

Discrete Exponent Function (11/14)

Notice that since G_5 is a subgroup of Z_{11}^* the multiplication operations in it are performed **mod** 11.

The exponent table shows that all elements {3, 4, 5, 9} are the generators in G_5.

Notice also that for all $\gamma \in \{3, 4, 5, 9\}$ their exponents 0 and 5 yields the same result, i.e.

$$\gamma^0 = \gamma^5 = 1 \text{ mod } 11.$$

This means that exponents of generators γ are computed **mod** 5.

This property makes the usage of modular groups of prime order q valuable in cryptography since they provide a higher-level security based on the stronger assumptions we will mention later.

Therefore, in many cases instead the group Z_p^* defined by the prime (not necessarily strong prime) number p the subgroup of prime order G_q in Z_p^* is used.

In this case if p is strong prime, then generator γ in G_q can be found by random search satisfying the following conditions

$$\gamma^q = 1 \textbf{ mod } p \text{ and } \gamma^2 \neq 1 \textbf{ mod } p.$$

Analogously in this generalized case this means that exponents of generators γ are computed **mod** q. In our modeling we will use group Z_p^* instead of G_q for simplicity.

Discrete Exponent Function (12/14)

Let as above p=11 and is strong prime and generator we choose g=7 from the set Γ={2, 6, 7, 8}.
Public Parameters are **PP**=(**11**,**7**), Then **DEF**$_g$(x) = **DEF**$_7$(x) is defined in the following way:

$$\mathbf{DEF}_7(x) = 7^x \mathbf{\ mod\ } 11 = a;$$

DEF$_7$(x) provides the following 1-to-1 mapping, displayed in the table below.

x	0	1	2	3	4	5	6	7	8	9	*10*	*11*	*12*	*13*	*14*
7^x **mod** p = a	**1**	**7**	**5**	**2**	**3**	**10**	**4**	**6**	**9**	**8**	*1*	*7*	*5*	*2*	*3*

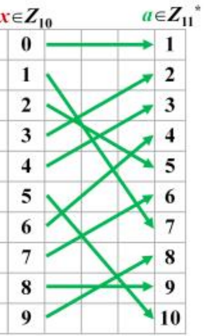
You can see that a values are repeating when x = 10, 11, 12, 13, 14, etc. since exponents are reduced **mod** 10 due to *Fermat little theorem*.
The illustration why 7^x **mod** p values are repeating when x = 10, 11, 12, 13, 14, etc. is presented in computations below:

10 mod 10 = 0; $7^{10} = 7^0 =$ 1 mod 11 = 1.
11 mod 10 = 1; $7^{11} = 7^1 =$ 7 mod 11 = 7.
12 mod 10 = 2; $7^{12} = 7^2 =$ 49 mod 11 = 5.
13 mod 10 = 3; $7^{13} = 7^3 =$ 343 mod 11 = 2.
14 mod 10 = 4; $7^{14} = 7^4 =$ 2401 mod 11 = 3.
etc.

Discrete Exponent Function (13/14)

For illustration of 1-to-1 mapping of **DEF**$_7$(x) we perform the following step-by-step computations.

$7^0 = 1$ **mod** 11
$7^1 = 7$ **mod** 11
$7^2 = 5$ **mod** 11
$7^3 = 2$ **mod** 11
$7^4 = 3$ **mod** 11
$7^5 = 10$ **mod** 11
$7^6 = 4$ **mod** 11
$7^7 = 6$ **mod** 11
$7^8 = 9$ **mod** 11
$7^9 = 8$ **mod** 11



It is seen that one value of x is mapped to one value of a.

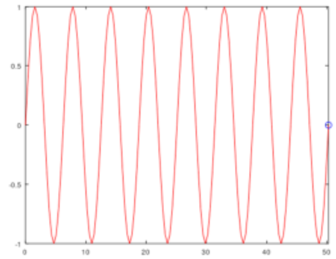
Discrete Exponent Function (14/14)

But the most in interesting think is that **DEF** is behaving like a *pseudorandom function*.

It is a main reason why this function is used in cryptography - classical cryptography.

To better understand the pseudorandom behaviour of **DEF** we compare the graph of "regular" **sine** function with "pseudorandom" **DEF** using Octave software.

```
>> p128sin
xrange = 16 * pi;
step = xrange/128;
x = 0:step:xrange;
y = sin(x);
comet(x, y)
```



```
>> p128def
p = 127;
g = 23;
x = 0:p-1;
a = mod_expv(g, x, p);
comet(x, a)
```